Announcements

1) Notes revised online (S is always on the outside)

2) Exam 3 Wednesday next week

Eigenvalues:

Suppose v is an eigenvector for λ, associated to A.

Then $Av = \lambda v$.

If c is any real number,

$$\begin{aligned} A(cv) &= cAv \quad \text{(linearity)} \\ &= c\lambda v \quad \text{(v eigenvector)} \\ &= \lambda(cv) \end{aligned}$$

So cv is an eigenvector for A if v is.

In particular, $-v$ is an eigenvector if v is.

Moreover, if $Av = \lambda v$ and $Aw = \lambda w$, then

$$\begin{aligned} A(v+w) &= Av + Aw \quad \text{(linearity)} \\ &= \lambda v + \lambda w \\ &= \lambda(v+w) \end{aligned}$$

This completes the proof that the subset of $\mathbb{R}^n$ consisting of all v with $Av = \lambda v$ for a particular λ is a subspace of $\mathbb{R}^n$!

It is called the eigenspace of A associated to λ.

Matrix Norm

(not in book!)

This will work for any norm on $\mathbb{R}^n$; we'll use $\|\cdot\|_2$.

Definition: Let A be an $m \times n$ matrix. Then we define the **norm** of A to be the smallest number c with

$$\boxed{\|Av\|_2 \leq c\|v\|_2}$$

for all v in $\mathbb{R}^n$.

We denote this value by $\|A\|$.

How to find it?

$$\|A\|^2 = \|A^tA\|.$$

A^tA is symmetric

since $(A^tA)^t = A^t(A^t)^t$

$$= A^tA.$$

So there is an orthogonal matrix S and a diagonal matrix D with $A^tA = SDS^{-1} = SDS^t$

But $\|SDS^t\|$

$= \|D\|$ if S is orthogonal, and $\|D\|$ is equal to the largest number on the diagonal (in absolute value).

So to find $\|A\|$: find the largest eigenvalue of A^tA, take square root.

Example 1 (2×2)

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}.$$

Find $\|A\|$.

Calculate $A^t A$

$$= \begin{bmatrix} 1 & -3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -16 \\ -16 & 40 \end{bmatrix}$$

The eigenvalues of A^tA are $\lambda_1 = 25 + \sqrt{481}$

$\lambda_2 = 25 - \sqrt{481}$

$\lambda_1 > \lambda_2$, so

$$\boxed{\|A\| = \sqrt{\lambda_1} = \sqrt{25 + \sqrt{481}}}$$

Example 2: (3×3)

$$A = \begin{bmatrix} 0 & 5 & -8 & 1 \\ 2 & 0 & 4 & 0 \end{bmatrix}$$

Find $\|A\|$.

Calculate A^tA

$$= \begin{bmatrix} 0 & 2 \\ 5 & 0 \\ -8 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 & -8 & 1 \\ 2 & 0 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 8 & 0 \\ 0 & 25 & -40 & 5 \\ 8 & -40 & 80 & -8 \\ 0 & 5 & -8 & 1 \end{bmatrix}$$

The eigenvalues of

A^tA are

$$\lambda_1 = 55 + \sqrt{2249}$$

$$\lambda_2 = 55 - \sqrt{2249}$$

$$\lambda_3 = \lambda_4 = 0$$

The biggest eigenvalue is λ_1,

so $$\boxed{\begin{aligned} \|A\| &= \sqrt{\lambda_1} \\ &= \sqrt{55+\sqrt{2249}} \end{aligned}}$$

Wolfram Alpha

Type in "norm of A".

this gives the norm!

Property of Transpose:

Why is $\|A\|^2 = \|A^tA\|$?

Polar Decomposition

(also not in text)

Orthogonal Diagonalization

We know that if $A = A^t$, we can diagonalize A via an orthogonal matrix. This is not possible in general – what is the best we can say?

Square Roots

If A satisfies

$(Av)\cdot v \geq 0$ for all v

(positive semi-definite),

then A is symmetric and all eigenvalues are non-negative.

If $A = SDS^t$

where the diagonal entries of D are non-negative, let $\sqrt{D}$ be the diagonal matrix given by taking square roots of the entries of D.

We define

$$\boxed{\sqrt{A} = S\sqrt{D}S^t}$$

$$(\sqrt{A})^2 = (S\sqrt{D}S^t)(S\sqrt{D}S^t)$$

$$= S\sqrt{D}\sqrt{D}S^t \quad (S \text{ orthogonal})$$

$$= SDS^t = A.$$

Definition: If A is in $M_n(\mathbb{R})$, define

$$\boxed{|A| = \sqrt{A^t A}}$$

We say A has polar decomposition if there is an orthogonal matrix W with

$$\boxed{A = W|A|}$$

Best possible scenario

A invertible :

Then $\boxed{w = A(|A|^{-1})}$

Example 3: (2×2)

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}.$$

Find the polar decomposition of A.

$$A^t A = \begin{bmatrix} 10 & -16 \\ -16 & 40 \end{bmatrix}$$

$$= S D S^{-1}$$

Where

$$D = \begin{bmatrix} 25+\sqrt{481} & 0 \\ 0 & 25-\sqrt{481} \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{16}\left(15-\sqrt{481}\right) & 1 \\ 1 & -\frac{1}{16}\left(15-\sqrt{481}\right) \end{bmatrix}$$

(we can choose S symmetric)

Then

$$|A| = \sqrt{A^t A}$$

$$= S \sqrt{D}\, S^{-1}$$

$$= S \begin{bmatrix} \sqrt{25+\sqrt{481}} & 0 \\ 0 & \sqrt{25-\sqrt{481}} \end{bmatrix} S^{-1}$$

= a monstrosity - see next page

$$\frac{\begin{bmatrix} -128\sqrt{25-\sqrt{481}} + \sqrt{25+\sqrt{481}}\left(-335+15\sqrt{481}\right) & -8\left(-15+\sqrt{481}\right)\left(\sqrt{25-\sqrt{481}} - \sqrt{25+\sqrt{481}}\right) \\ -8\left(-15+\sqrt{481}\right)\left(\sqrt{25-\sqrt{481}} - \sqrt{25+\sqrt{481}}\right) & -353\sqrt{25-\sqrt{481}} + 15\sqrt{481\left(25-\sqrt{481}\right)} - 128\sqrt{25+\sqrt{481}} \end{bmatrix}}{481 - 15\sqrt{481}}$$

Since the eigenvalues of $|A|$ are nonzero, $|A|$ is invertible.

Then

$$w = A|A|^{-1}$$

$$= \text{Some greater monstrosity.}$$

Let's try this with a nice example!

Example 4: (3x3)

Let

$$A = \frac{1}{2}\begin{bmatrix} 5 & 3 & -5 \\ -3 & 5 & 5 \\ 0 & 0 & 18 \end{bmatrix}$$

Find the polar decomposition of A

$$A^tA$$

$$= \begin{bmatrix} 17/2 & -15/2 & -10 \\ -15/2 & 17/2 & 10 \\ -10 & 10 & 187/2 \end{bmatrix}$$

$$= SDS^{-1} \quad \text{where}$$

$$S = \begin{bmatrix} 1 & 4 & -\frac{1}{8} \\ 1 & -4 & \frac{1}{8} \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 27/2 & 0 \\ 0 & 0 & 96 \end{bmatrix}$$

Then

$$|A| = \sqrt{A^t A}$$

$$= S\sqrt{D}\,S^{-1}$$

$$= \frac{1}{66}\begin{bmatrix} 33+52\sqrt{6} & 33-52\sqrt{6} & -60\sqrt{2/3} \\ 33-52\sqrt{6} & 33+52\sqrt{6} & 60\sqrt{2/3} \\ -60\sqrt{2/3} & 60\sqrt{2/3} & 259\sqrt{6} \end{bmatrix}$$

Since $|A|$ does not have zero as an eigenvalue, $|A|$ is invertible

Then

$$W = A\,|A|^{-1}$$

$$= \frac{1}{66}\begin{bmatrix} 132+\frac{53}{4\sqrt{6}} & 132-\frac{53}{4\sqrt{6}} & -20\sqrt{\frac{2}{3}} \\ 33-13\sqrt{6} & 33+13\sqrt{6} & 5\sqrt{6} \\ 5\sqrt{6} & -5\sqrt{6} & 78\sqrt{\frac{2}{3}} \end{bmatrix}$$

What if A is not invertible?